Physica A 445 (2016) 240–255

Contents lists available at ScienceDirect

Physica A

journal homepage: www.elsevier.com/locate/physa

Non-Euclidean-normed Statistical Mechanics

George Livadiotis

Southwest Research Institute, San Antonio, TX-78238, USA

HIGHLIGHTS

- Statistical Mechanics is generalized in the framework of non-Euclidean metrics induced by $\mathcal{L}^p$ norms.
- The non-Euclidean-normed canonical distribution for a power-law energy density states is formed.
- The range of possible values of the q-index, which depends on the value "p" of the $\mathcal{L}^p$-norm, is derived.
- The physical temperature coincides with the kinetically defined temperature.
- The new Statistical Mechanics follows the standard classical formalisms of thermodynamics.

ARTICLE INFO

Article history:
Received 29 August 2015
Received in revised form 22 October 2015
Available online 7 November 2015

Keywords:
Non-Euclidean norms
Generalized means
Non-extensive Statistical Mechanics

ABSTRACT

This analysis introduces a possible generalization of Statistical Mechanics within the framework of non-Euclidean metrics induced by the $\mathcal{L}^p$ norms. The internal energy is interpreted by the non-Euclidean $\mathcal{L}^p$-normed expectation value of a given energy spectrum. The presented non-Euclidean adaptation of Statistical Mechanics involves finding the stationary probability distribution in the Canonical Ensemble by maximizing the Boltzmann–Gibbs and Tsallis entropy under the constraint of internal energy. The derived non-Euclidean Canonical probability distributions are respectively given by an exponential, and by a q-deformed exponential, of a power-law dependence on energy states. The case of the continuous energy spectrum is thoroughly examined. The Canonical probability distribution is analytically calculated for a power-law density of energy. The relevant non-Euclidean-normed kappa distribution is also derived. This analysis exposes the possible values of the q- or κ-indices, which are strictly limited to certain ranges, depending on the given $\mathcal{L}^p$-norm. The equipartition of energy in each degree of freedom and the extensivity of the internal energy, are also shown. Surprisingly, the physical temperature coincides with the kinetically defined temperature, similar to the Euclidean case. Finally, the connection with thermodynamics arises through the well-known standard classical formalisms.

1. Introduction

The Euclidean $\mathcal{L}^2$-normed mean can be derived by minimizing the sum of the square ($\mathcal{L}^2$) deviations, that is the Euclidean variance. In particular, given the elements $\{y_i\}_{i=1}^N$, $y_i \in D_y \subseteq \Re, \forall i = 1, \ldots, N$, then the Total Square Deviations $TSD(\{y_i\}_{i=1}^N; \alpha)^2 = \sum_{i=1}^N |y_i - \alpha|^2$ are minimized for the optimal approximating finding value $\alpha^* = \mu_2$, that is the Euclidean mean, given by $\sum_{i=1}^N |y_i - \mu_2| \mathrm{sign}(y_i - \mu_2) = 0$ ($\Leftrightarrow \mu_2 = \frac{1}{N}\sum_{i=1}^N y_i$). In a similar way, the non-Euclidean $\mathcal{L}^p$-normed means μ_p ($p \geq 1$) were deduced in Ref. [1], by minimizing the sum of the $\mathcal{L}^p$-normed deviations, or, Total p-Deviations,

$$TD_p(\{y_i\}_{i=1}^N; \alpha; p)^p = \sum_{i=1}^N |y_i - \alpha|^p. \tag{1}$$

E-mail addresses: glivadiotis@swri.edu, glivad@phys.uoa.gr.

http://dx.doi.org/10.1016/j.physa.2015.11.002

The optimization leads to the normal equation

$$\sum_{i=1}^{N} |y_i - \mu_p|^{p-1} \mathrm{sign}(y_i - \mu_p) = 0, \tag{2}$$

from which the optimal parameter $\alpha^* = \mu_p$, that is the non-Euclidean $\mathcal{L}^p$-normed mean, is derived as an implicit expression of p. (See also: Refs. [2,3].)

This generalization obeys in a formal scheme of means characterization, given by the univalued, N-multivariable function $M(\{y_i\}_{i=1}^N)$, fulfilling the three preconditions: (i) Continuity; (ii) Internness: $\mathrm{Min}(\{y_i\}_{i=1}^N) \leq M(\{y_i\}_{i=1}^N) \leq \mathrm{Max}(\{y_i\}_{i=1}^N)$; (iii) Symmetry: For $y_i \to y_{j_i}, \forall\, i = 1, \ldots, N$: $\{y_i\}_{i=1}^N = \{y_{j_i}\}_{i=1}^N$, then $M(\{y_i\}_{i=1}^N) = M(\{y_{j_i}\}_{i=1}^N)$.

Furthermore, given the spectrum of y-values $\{y_k\}_{k=1}^W$, associated with the possibilities $\{p_k\}_{k=1}^W$, the concept of expectation value is generalized to the non-Euclidean adaptation $\langle y \rangle_p$ that is implicitly expressed by

$$\sum_{k=1}^{W} p_k |y_k - \mu_p|^{p-1} \mathrm{sign}(y_k - \mu_p) = 0, \tag{3}$$

where the classical Euclidean expectation value is recovered for $p = 2$, i.e., $\langle y \rangle_2 = \sum_{k=1}^W p_k y_k$. According to this, the internal energy U_p of a system that characterizes by a discrete energy spectrum $\{\varepsilon_k\}_{k=1}^W$, associated with the discrete probability distribution $\{p_k\}_{k=1}^W$, is interpreted by the non-Euclidean $\mathcal{L}^p$-normed expectation value, that is implicitly given by

$$\sum_{k=1}^{W} p_k |\varepsilon_k - U_p|^{p-1} \mathrm{sign}(\varepsilon_k - U_p) = 0. \tag{4}$$

This is written in terms of the non-Euclidean norm operator $\hat{\mathcal{L}}_p$, defined by

$$\hat{\mathcal{L}}_p(\varepsilon_k) = \frac{|\varepsilon_k - U_p|^{p-1} \mathrm{sign}(\varepsilon_k - U_p)}{(p-1)\phi_p} + U_p, \tag{5}$$

namely,

$$U_p = \langle \hat{\mathcal{L}}_p(\varepsilon) \rangle_2 = \sum_{k=1}^{W} p_k \hat{\mathcal{L}}_p(\varepsilon_k), \quad \text{or,} \quad \langle \hat{\mathcal{L}}_p(\varepsilon - U_p) \rangle_2 = 0. \tag{6}$$

The argument ϕ_p is defined by

$$\phi_p \equiv \sum_{k=1}^{W} p_k |\varepsilon_k - U_p|^{p-2}, \tag{7}$$

that is the appropriate expression for deducing the zero-mean of the derivative of $\{\hat{\mathcal{L}}_p(\varepsilon_k)\}_{k=1}^W$ with respect to a given parameter β, for which $p_k = p_k(\{\varepsilon_{k'}\}_{k'=1}^W; \beta), \forall\, k = 1, \ldots, W$, and $U_p = U_p(\{\varepsilon_k\}_{k=1}^W; \beta)$, i.e.,

$$0 = \sum_{k=1}^{W} p_k \frac{\partial}{\partial \beta} \hat{\mathcal{L}}_p(\varepsilon_k), \quad \text{or,} \quad \frac{\partial}{\partial \beta} \langle \varepsilon \rangle_p = \sum_{k=1}^{W} \frac{\partial p_k}{\partial \beta} \hat{\mathcal{L}}_p(\varepsilon_k). \tag{8}$$

In Ref. [1] the argument ϕ_p was the key-point for extracting the exact expression of the $\mathcal{L}^p$-normed variance. In addition, the zero-mean property, given in Eq. (8), is fulfilled if and only if the argument ϕ_p has the specific expression given in Eq. (7). Only then, the Canonical probability distribution can be derived. If ϕ_p were expressed by any other formulation, after the extremization of entropy in the Canonical Ensemble, we would not be able to solve in terms of the probability. Moreover, the zero-mean property helps to connect the non-Euclidean Statistical Mechanics with Thermodynamics. (Regarding the expression of ϕ_p and the importance of the zero-mean property, see Ref. [1].)

As another point of view, [4] generalized the ordinary (Euclidean) expectation value $\langle \varepsilon \rangle_2 = \sum_{k=1}^W p_k \varepsilon_k$ to the escort expectation value, i.e.,

$$\langle \varepsilon \rangle_q = \sum_{k=1}^{W} P_k \varepsilon_k, \tag{9}$$

where the escort probability distribution $\{P_k\}_{k=1}^W$ is constructed via the ordinary probability distribution $\{p_k\}_{k=1}^W$ and the duality relation [5–7]

$$P_k \equiv {p_k}^q \Big/ \sum_{k'=1}^{W} {p_{k'}}^q, \qquad p_k = {P_k}^{1/q} \Big/ \sum_{k'=1}^{W} {P_{k'}}^{1/q}. \tag{10}$$

In order to derive the stationary probability distribution of the energy spectrum $\{\varepsilon_k\}_{k=1}^W$ in the Canonical Ensemble, the Tsallis entropy is maximized under the constraint of fixed internal energy, [8], i.e.,

$$S^{\mathcal{TS}}(\{p_k\}_{k=1}^W; q) = \sum_{k=1}^W p_k \ln_q\left(\frac{1}{p_k}\right) = \frac{1 - \sum\limits_{k=1}^W {p_k}^q}{q-1}, \tag{11}$$

recovering the BG entropy $S^{\mathcal{BG}}(\{p_k\}_{k=1}^W) = \sum_{k=1}^W p_k \ln(\frac{1}{p_k})$ for $q \to 1$. The internal energy was expressed, first, by the ordinary probability distribution $U = \sum_{k=1}^W p_k \varepsilon_k$ [8], while thereafter, by considering the escort probability distribution $U_q = \sum_{k=1}^W P_k \varepsilon_k$ [4]. By utilizing the escort probabilities, [4] succeeded to recover (i) the extensivity of the internal energy among independent subsystems, and (ii) the invariancy of the Canonical probability distribution $\{p_k\}_{k=1}^W$ for an arbitrary ground-level energy. Hence,

$$p_k(\{\varepsilon_{k'}\}_{k'=1}^W; q) = \frac{1}{\mathcal{Z}_q} \exp_q(-\beta_q U_q) \exp_q[-\beta_q(\varepsilon_k - U_q)], \tag{12}$$

where

$$\mathcal{Z}_q = \exp_q(-\beta_q U_q) \sum_{k=1}^W \exp_q[-\beta_q(\varepsilon_k - U_q)], \tag{13}$$

reads the partition function in Tsallis non-extensive Statistical Mechanics. The function $\exp_q(u) = [1 + (1-q)u]_+^{\frac{1}{1-q}}$ is the q-deformed exponential, while its inverse, $\ln_q(u) = \frac{1}{q-1}(1-u^{1-q})$, with $\exp_q[\ln_q(u)] = \ln_q[\exp_q(u)] = u$, is the q-deformed logarithm [9,10]. (The symbol $[x]_+$ denotes the cut-off condition: $[x]_+ = x$, if $x \geq 0$, and $[x]_+ = 0$, if $x \leq 0$.) In addition, we set

$$\phi_q \equiv \sum_{k=1}^W {p_k}^q, \qquad \beta_q \equiv \beta/\phi_q, \qquad T_q \equiv \phi_q T, \tag{14}$$

with $\beta_q \equiv \frac{1}{k_B T_q}$ and $\beta \equiv \frac{1}{k_B T}$. The argument T_q reads the physical temperature [11,12] that generalizes the zero-th law of thermodynamics (that two bodies in thermal equilibrium with a third, are also in thermal equilibrium with each other). Eq. (14) shows that the physical temperature T_q is connected with the "Lagrangian temperature" T (the one related to the second Lagrangian multiplier), via the argument ϕ_q [13]. In general, T_q and T differ from each other, except at equilibrium ($q \to 1$). As it is shown in Ref. [14] for the case of continuous energy states, the physical temperature T_q is the actual temperature for stationary states out of equilibrium, instead of the Lagrangian T. Hence, T_q does not depend on the q-index, but both T_q and q are physical independent parameters of the system. (Here, the subscript q does not denote dependence of T_q on q but it is simply for showing the connection with Tsallis statistical theory; others, use different subscripts such "phys", or no subscript at all since it is the real temperature, in contrast to the Lagrangian temperature-like parameter that may take a different symbol or subscript, i.e., T_L. Nonetheless, we will keep the original symbolism of T_q in this paper.) The primary importance of T_q is due to the fact that T_q coincides with the kinetically defined temperature T_K as extracted by the second statistical moment of the Tsallis-like Maxwellian distribution of velocities. (This is the probability distribution as given in Eq. (12), where we express the kinetic energy in terms of velocity.) Refs. [15,16] showed that the kinetic definition of temperature does not absolutely satisfy the zero-th law of thermodynamics out of equilibrium. However, the physical temperature is obtained in accordance with the generalized zero-th law [17–19]. Therefore, all the advantages of a kinetically defined temperature, in contrast to other definitions [20], can be ascribed to T_q. In addition, the inconsistencies of the BG kinetic definition of temperature in regards to the zero-th law of thermodynamics [15,16] are fully recovered, since the origin of T_q establishes the generalized zero-th law. The analysis will show that the interpretation of the physical temperature T_q as a kinetically defined temperature is preserved even in the non-Euclidean adaptation of Tsallis Statistical Mechanics. This significant feature of T_q reads the equipartition of energy in each degree of freedom, and the extensivity of the internal energy in the continuous energy spectrum (see also Refs. [21–23]).

Note that the probability distribution in Eq. (12) can be written in a simpler and more obvious form in terms of the auxiliary partition function defined by $\mathcal{Z}'_q = \sum_{k=1}^W \exp_q[-\beta_q(\varepsilon_k - U_q)]$, or even by $\mathcal{Z}''_q = \sum_{k=1}^W \exp_q(-\beta'_q \varepsilon_k)$ with $\beta'_q = \beta_q[1 + (1-q)\beta_q U_q]^{-1}$. However, the connection with thermodynamics is succeeded by utilizing $\mathcal{Z}_q$, instead of $\mathcal{Z}'_q$ [24,25].

The classical exponential probability distribution recovers for $q \to 1$, i.e.,

$$p_k^{\mathcal{BG}}(\{\varepsilon_{k'}\}_{k'=1}^W) = \frac{1}{\mathcal{Z}^{\mathcal{BG}}} \exp(-\beta\varepsilon_k), \tag{15}$$

where

$$\mathcal{Z}^{\mathcal{BG}} = \mathcal{Z}_{q=1} = \sum_{k=1}^W \exp(-\beta\varepsilon_k), \tag{16}$$

reads the classical BG partition function. We shall see how the Canonical probability distribution, within the framework either of BG or of Tsallis Statistical Mechanics, can be generalized by considering the non-Euclidean metrics induced by $\mathcal{L}^p$-norms. In the case of the non-Euclidean Tsallis Statistical Mechanics, the non-Euclidean norm operator $\hat{\mathcal{L}}_{(p,q)}$ is given by

$$\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k) = \frac{|\varepsilon_k - U_{(p,q)}|^{p-1}\mathrm{sign}(\varepsilon_k - U_{(p,q)})}{(p-1)\phi_{(p,q)}} + U_{(p,q)}. \tag{17}$$

The internal energy $U_{(p,q)}$ is represented by the $\mathcal{L}^p$-normed escort, or simply, (p, q)-expectation value of $\{\varepsilon_k\}_{k=1}^W$, that is $U_{(p,q)} = \langle\varepsilon\rangle_{(p,q)}$, instead of the Euclidean $U_{(2,1)} = \langle\varepsilon\rangle_{(2,1)} = \langle\varepsilon\rangle$ of BG statistics, or of the Euclidean escort $U_{(2,q)} = \langle\varepsilon\rangle_{(2,q)} = \langle\varepsilon\rangle_q$ of Tsallis statistics. Namely,

$$\sum_{k=1}^W P_k|\varepsilon_k - U_{(p,q)}|^{p-1}\mathrm{sign}(\varepsilon_k - U_{(p,q)}) = 0. \tag{18}$$

This is written in terms of the non-Euclidean norm operator $\hat{\mathcal{L}}_{(p,q)}$ as follows

$$U_{(p,q)} = \langle\hat{\mathcal{L}}_{(p,q)}(\varepsilon)\rangle_{(2,q)} = \sum_{k=1}^W P_k\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k), \quad \text{or,} \quad \langle\hat{\mathcal{L}}_{(p,q)}(\varepsilon - U_{(p,q)})\rangle_{(2,q)} = 0, \tag{19}$$

where the subindex $(2, q)$, or simply q, indicates the non-Euclidean escort mean of $\{\varepsilon_k\}_{k=1}^W$, that is the Euclidean escort mean of $\{\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k)\}_{k=1}^W$, i.e., estimated by utilizing the escort probability distribution. Moreover, the relevant argument ϕ_p, given in Eq. (7), is replaced by the factor $\phi_{(p,q)} \equiv \sum_{k=1}^W P_k|y_k - \langle y\rangle_p|^{p-2}$, while, given $\phi_q \equiv \sum_{k=1}^W {p_k}^q$, we can also define the convenient argument $\varphi_{(p,q)} \equiv \phi_q\phi_{(p,q)} = \sum_{k=1}^W {p_k}^q|y_k - \langle y\rangle_p|^{p-2}$.

The paper is organized as follows: In Section 2, the non-Euclidean adaptations of BG and Tsallis Canonical probability distribution are derived. This is attained by extremizing the BG and Tsallis entropic formulations, respectively, under the constraint of internal energy, being interpreted by the $\mathcal{L}^p$-normed expectation value of the spectrum of energy states. The extracted Canonical probability distributions are expressed in terms of the non-Euclidean operator that is acting upon the energy states. Then, these probability distributions are modified so that to be feasible to be explicitly expressed in terms of the energy spectrum and the temperature. Furthermore, the case of the continuous energy spectrum is thoroughly examined in Section 3. In particular, in the case of a power-law density of energy states, the non-Euclidean Canonical probability is found analytically for both the non-Euclidean adaptations of BG and of Tsallis Statistical Mechanics. The permissible values of the q-index in regards to the given p-norm, are exposed. The equipartition of energy in each degree of freedom, as well as the extensivity of internal energy, were shown, where the physical temperature T_q coincides with the kinetically defined temperature T_K. In Section 4, it is shown that the connection with thermodynamics arises through the well-known classical formalisms. In these relations, the partition function and the internal energy are replaced by the respective non-Euclidean one. The relevant relations that connect Tsallis Statistical Mechanics with Thermodynamics are the same as in the Euclidean case, where the formalism of the q-deformed logarithm is considered. Finally, Section 5 summarizes the conclusions.

2. Canonical ensemble

2.1. Derivation of Canonical probability distribution

The non-Euclidean adaptation of Tsallis thermal equilibrium probability distribution is derived by following along the famous Gibbs' path (e.g., see Refs. [8,24]), where the entropy $S^{\mathcal{TS}}$ (Eq. (11)) is extremized under the constraints (1) $\sum_{k=1}^W p_k = 1$ (normalization), and (2) $\sum_{k=1}^W P_k\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k) = U_{(p,q)}$ (internal energy), that is by optimizing the functional

$$G_{(p,q)}(\{p_k\}_{k=1}^W; p, q) = S^{\mathcal{TS}}(\{p_k\}_{k=1}^W; q) + \lambda_1\sum_{k=1}^W p_k + \lambda_2\langle\hat{\mathcal{L}}_{(p,q)}(\varepsilon)\rangle_q(\{p_k\}_{k=1}^W; p, q), \tag{20}$$

where λ_1, λ_2 are the Lagrange multipliers. Hence,

$$0 = \frac{\partial G_{(p,q)}}{\partial p_j} = -\frac{q}{q-1}p_j^{q-1} + \lambda_1 + \lambda_2\left[\sum_{k=1}^W \frac{\partial P_k}{\partial p_j}\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k) + \sum_{k=1}^W P_k\frac{\partial}{\partial p_j}\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k)\right].$$

The implicit expression of $\langle\hat{\mathcal{L}}_{(p,q)}(\varepsilon)\rangle_q$ in terms of $U_{(p,q)}$ should make this calculation step impossible to be solved. Fortunately, the specific expression of the argument ϕ_p, as given in Eq. (7), that leads to the zero-mean of $\hat{\mathcal{L}}_{(p,q)}(\varepsilon_j)$, i.e., $\sum_{k=1}^W p_k\frac{\partial}{\partial p_j}\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k) = 0$ (given in Eq. (17) for $\beta \to p_j$), allows the expression of probability p_j to be unfettered and given in terms of $\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k - U_{(p,q)})$. Indeed,

$$p_j \propto \left[1 + \frac{\lambda_2}{\phi_q}(1-q)\hat{\mathcal{L}}_{(p,q)}(\varepsilon_j - U_{(p,q)})\right]^{\frac{1}{1-q}},$$

because of $\frac{\partial P_k}{\partial p_j} = q p_j^{q-1} \phi_q^{-1} (\delta_{kj} - P_k)$ that leads to

$$\sum_{k=1}^{W} \frac{\partial P_k}{\partial p_j} \hat{\mathcal{L}}_{(p,q)}(\varepsilon_k) = q p_j^{q-1} \phi_q^{-1} \cdot \left[\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k) - U_{(p,q)} \right] = q p_j^{q-1} \phi_q^{-1} \cdot \hat{\mathcal{L}}_{(p,q)}(\varepsilon_k - U_{(p,q)}).$$

Setting $j \to k$, $-\lambda_2 \equiv \beta \equiv \beta_q \phi_q$, and reestablishing λ_1, we conclude in

$$p_k(\{\varepsilon_{k'}\}_{k'=1}^{W}; \beta_q; p, q) = \frac{1}{\mathcal{Z}_{(p,q)}} \exp_q(-\beta_q U_{(p,q)}) \exp_q\left[-\beta_q \hat{\mathcal{L}}_{(p,q)}(\varepsilon_k - U_{(p,q)}) \right], \tag{21}$$

where the generalized partition function is given by

$$\mathcal{Z}_{(p,q)} = \exp_q(-\beta_q U_{(p,q)}) \sum_{k=1}^{W} \exp_q\left[-\beta_q \hat{\mathcal{L}}_{(p,q)}(\varepsilon_k - U_{(p,q)}) \right]. \tag{22}$$

The non-Euclidean Tsallis probability distribution recovers in its Euclidean version, given in Eqs. (12), (13), by setting $p = 2$. Notice that the only difference between Eqs. (12), (21) is the replacement of $(\varepsilon_k - U_q)$ with $\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k - U_{(p,q)})$, or equivalently, the replacement of ε_k with $\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k)$. This remark is essential and verifies that by replacing ε_k with $\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k)$ within an Euclidean formulation we can automatically retrieve the respective non-Euclidean one.

We proceed in deriving the non-Euclidean BG probability distribution. By extremizing the entropy $S_{\mathcal{BG}}$ under the constraints (1) $\sum_{k=1}^{W} p_k = 1$ and (2) $\sum_{k=1}^{W} p_k \hat{\mathcal{L}}_p(\varepsilon_k) = U_p$, i.e., by extremizing the functional

$$G_p(\{p_k\}_{k=1}^{W}; p) = S^{\mathcal{BG}}(\{p_k\}_{k=1}^{W}) + \lambda_1 \sum_{k=1}^{W} p_k + \lambda_2 \langle \hat{\mathcal{L}}_p(\varepsilon) \rangle_2 \left(\{p_k\}_{k=1}^{W}; p\right), \tag{23}$$

we obtain

$$0 = \frac{\partial G_p}{\partial p_j} = -\ln p_j - 1 + \lambda_1 + \lambda_2 \left[\sum_{k=1}^{W} \delta_{kj} \hat{\mathcal{L}}_p(\varepsilon_k) + \sum_{k=1}^{W} p_k \frac{\partial}{\partial p_j} \hat{\mathcal{L}}_p(\varepsilon_k) \right],$$

or $0 = -\ln p_j - 1 + \lambda_1 + \lambda_2 \hat{\mathcal{L}}_p(\varepsilon_k)$, and by considering $j \to k$, $-\lambda_2 = \beta$, and rewriting λ_1, we arrive at

$$p_k(\{\varepsilon_{k'}\}_{k'=1}^{W}; \beta; p) = \frac{1}{\mathcal{Z}_p} \exp\left[-\beta \hat{\mathcal{L}}_p(\varepsilon_k) \right], \tag{24}$$

where the non-Euclidean BG partition function is given by

$$\mathcal{Z}_p = \sum_{k=1}^{W} \exp\left[-\beta \hat{\mathcal{L}}_p(\varepsilon_k) \right]. \tag{25}$$

The non-Euclidean BG probability distribution recovers in its classical Euclidean form, given in Eqs. (15), (16), by setting $p = 2$. The non-extensive adaptation of Eqs. (21), (22) recovers in Eqs. (24), (25) by setting $q \to 1$. Once again, we verify that the non-Euclidean formulations can be automatically derived by the respective Euclidean equations (15), (16), by replacing $\{\varepsilon_k\}_{k=1}^{W}$ with $\{\hat{\mathcal{L}}_p(\varepsilon_k)\}_{k=1}^{W}$.

Finally, if the energy spectrum $\{\varepsilon_k\}_{k=1}^{W}$ is associated with the degeneracies $\{g_k\}_{k=1}^{W}$, then the Canonical probability distributions of non-Euclidean BG and Tsallis Statistical Mechanics are still given by Eqs. (24) and (21), respectively. However, the normalizations and the partition functions are rewritten accordingly, e.g., for the Tsallis statistics, we have, respectively,

$$\sum_{k=1}^{W} g_k p_k(\{\varepsilon_{k'}\}_{k'=1}^{W}; \beta_q; p, q) = 1, \qquad \sum_{k=1}^{W} g_k P_k(\{\varepsilon_{k'}\}_{k'=1}^{W}; \beta_q; p, q) = 1, \tag{26}$$

and

$$\mathcal{Z}_{(p,q)} = \exp_q(-\beta_q U_{(p,q)}) \sum_{k=1}^{W} g_k \exp_q\left[-\beta_q \hat{\mathcal{L}}_{(p,q)}(\varepsilon_k - U_{(p,q)}) \right]. \tag{27}$$

2.2. *Modified canonical probability distribution*

The probability distributions of Eqs. (24), (21) are not manageable, since they are expressed not only in terms of the energy spectrum $\{\varepsilon_k\}_{k=1}^{W}$ and the inverse temperature β (or β_q), but also in terms of the internal energy U_p (or $U_{(p,q)}$) and the argument ϕ_p (or $\phi_{(p,q)}$), which are quantities dependent implicitly on the probability distribution. In order to overcome this problem, we write the probability distribution in terms of an auxiliary parameter β_p (or $\beta_{(p,q)}$). Let start from the non-Euclidean BG case,

$$p_k(\{\varepsilon_{k'}\}_{k'=1}^{W}; \beta_p; p) = \frac{1}{A_p} \exp\left[-\frac{1}{p-1}(\beta_p|\varepsilon_k - U_p|)^{p-2}\beta_p(\varepsilon_k - U_p)\right], \tag{28}$$

where $|u|^{p-1}\mathrm{sign}(u) = |u|^{p-2}u$, $\forall\, u \in \Re$, and

$${\beta_p}^{p-1} \equiv \frac{\beta}{\phi_p}, \tag{29}$$

$$A_p \equiv \sum_{k=1}^{W} g_k \exp\left[-\frac{1}{p-1}(\beta_p|\varepsilon_k - U_p|)^{p-2}\beta_p(\varepsilon_k - U_p)\right]. \tag{30}$$

An analysis similar to that of $\beta \leftrightarrow \beta'_q$ transformation method [26], can be developed, in order to overcome the problem of the implicit expression of probabilities. In particular, we handle the auxiliary inverse temperature-like parameter β_p in non-Euclidean BG statistics, in similar way we treat the auxiliary inverse temperature-like parameter β'_q in Euclidean Tsallis statistics [27]. Indeed, first we express the internal energy U_p in terms of β_p, by solving numerically the non-Euclidean expectation value $\sum_{k=1}^{W} g_k p_k \hat{\mathcal{L}}_p(\varepsilon_k - U_p) = 0$,

$$\sum_{k=1}^{W} g_k \mathrm{e}^{-\frac{1}{p-1}(\beta_p|\varepsilon_k - U_p|)^{p-2}\beta_p(\varepsilon_k - U_p)} |\varepsilon_k - U_p|^{p-2}(\varepsilon_k - U_p) = 0. \tag{31}$$

Then, we express the ordinary inverse temperature β in terms of β_p as follows: The argument $\phi_p = \phi_p(\beta_p)$ is calculated as a function of β_p by

$$\phi_p \equiv \sum_{k=1}^{W} g_k p_k |\varepsilon_k - U_p|^{p-2} = \frac{\sum_{k=1}^{W} g_k\, \mathrm{e}^{-\frac{1}{p-1}(\beta_p|\varepsilon_k - U_p|)^{p-2}\beta_p(\varepsilon_k - U_p)}\, |\varepsilon_k - U_p|^{p-2}}{\sum_{k=1}^{W} g_k\, \mathrm{e}^{-\frac{1}{p-1}(\beta_p|\varepsilon_k - U_p|)^{p-2}\beta_p(\varepsilon_k - U_p)}}, \tag{32}$$

(where the internal energy U_p is given in terms of β_p by solving Eq. (31)). Then, we have $\beta(\beta_p) = {\beta_p}^{p-1}\phi_p(\beta_p)$, or $\beta_p = \beta_p(\beta)$, which together with $U_p(\beta_p)$, finally leads to $U_p = U_p[\beta_p(\beta)]$.

Moreover, within the framework of non-Euclidean Tsallis statistics, both the parameters β_p and β_q merge to a single one, $\beta_{(p,q)}$, namely,

$$p_k(\{\varepsilon_{k'}\}_{k'=1}^{W}; \beta_{(p,q)}; p, q) = \frac{1}{A_{(p,q)}} \exp_q\left[-\frac{1}{p-1}(\beta_{(p,q)}|\varepsilon_k - U_{(p,q)}|)^{p-2}\beta_{(p,q)}(\varepsilon_k - U_{(p,q)})\right], \tag{33}$$

where

$${\beta_{(p,q)}}^{p-1} \equiv \frac{\beta_q}{\phi_{(p,q)}} = \frac{\beta}{\varphi_{(p,q)}}, \tag{34}$$

$$A_{(p,q)} = \sum_{k=1}^{W} \exp_q\left[-\frac{1}{p-1}\Big(\beta_{(p,q)}|\varepsilon_k - U_{(p,q)}|\Big)^{p-2}\beta_{(p,q)}(\varepsilon_k - U_{(p,q)})\right]. \tag{35}$$

The internal energy $U_{(p,q)}$ is expressed in terms of $\beta_{(p,q)}$ by solving numerically the non-Euclidean expectation value $\sum_{k=1}^{W} P_k \hat{\mathcal{L}}_{(p,q)}(\varepsilon_k - U_{(p,q)}) = 0$, i.e.,

$$\sum_{k=1}^{W} g_k \exp_q{}^q\left[-\frac{1}{p-1}(\beta_{(p,q)}|\varepsilon_k - U_{(p,q)}|)^{p-2}\beta_{(p,q)}(\varepsilon_k - U_{(p,q)})\right]|\varepsilon_k - U_{(p,q)}|^{p-2}(\varepsilon_k - U_{(p,q)}) = 0, \tag{36}$$

where we denote $\exp_q{}^q(u) \equiv [\exp_q(u)]^q$. On the other hand, the Lagrangian inversetemperature β is expressed in terms of $\beta_{(p,q)}$ by using the function $\beta(\beta_{(p,q)}) = {\beta_{(p,q)}}^{p-1}\varphi_{(p,q)}(\beta_{(p,q)})$, where the function $\varphi_{(p,q)} = \varphi_{(p,q)}(\beta_{(p,q)})$ is derived from

$$\begin{aligned}\varphi_{(p,q)} &\equiv \sum_{k=1}^{W} g_k p_k^q |\varepsilon_k - U_{(p,q)}|^{p-2} \\ &= \frac{\sum_{k=1}^{W} g_k \exp_q{}^q\left[-\frac{1}{p-1}(\beta_{(p,q)}|\varepsilon_k - U_{(p,q)}|)^{p-2}\beta_{(p,q)}(\varepsilon_k - U_{(p,q)})\right]|\varepsilon_k - U_{(p,q)}|^{p-2}}{\left\{\sum_{k=1}^{W} g_k \exp_q\left[-\frac{1}{p-1}(\beta_{(p,q)}|\varepsilon_k - U_{(p,q)}|)^{p-2}\beta_{(p,q)}(\varepsilon_k - U_{(p,q)})\right]\right\}^q},\end{aligned} \tag{37}$$

(where the internal energy $U_{(p,q)}$ is given in terms of $\beta_{(p,q)}$ by solving Eq. (36)). Hence, we conclude in $U_{(p,q)} = U_{(p,q)}[\beta_{(p,q)}(\beta)]$. Notice that Eq. (36) leads to

$$\sum_{k=1}^{W} g_k \exp_q(z_k) = \sum_{k=1}^{W} g_k \exp_q{}^q(z_k), \tag{38}$$

where we set $z_k \equiv -\frac{1}{p-1}(\beta_{(p,q)}|\varepsilon_k - U_{(p,q)}|)^{p-1}\mathrm{sign}(\varepsilon_k - U_{(p,q)})$.

Finally, if we attend to express the internal energy in terms of the physical temperature T_q, or its inverse β_q, that is to find $U_{(p,q)} = U_{(p,q)}(\beta_q)$, then we work with $\phi_{(p,q)}$, instead of $\varphi_{(p,q)}$, given by

$$\begin{aligned}\phi_{(p,q)} &\equiv \sum_{k=1}^{W} g_k P_k |\varepsilon_k - U_{(p,q)}|^{p-2} \\ &= \frac{\sum_{k=1}^{W} g_k \exp_q{}^q\left[-\frac{1}{p-1}(\beta_{(p,q)}|\varepsilon_k - U_{(p,q)}|)^{p-2}\beta_{(p,q)}(\varepsilon_k - U_{(p,q)})\right]|\varepsilon_k - U_{(p,q)}|^{p-2}}{\sum_{k=1}^{W} g_k \exp_q\left[-\frac{1}{p-1}(\beta_{(p,q)}|\varepsilon_k - U_{(p,q)}|)^{p-2}\beta_{(p,q)}(\varepsilon_k - U_{(p,q)})\right]}.\end{aligned} \tag{39}$$

3. Continuous energy spectrum

3.1. Constant density of states

In the case of the continuous description of energy spectrum $\varepsilon \in [0, \infty)$, associated with a non-trivial density of states $g(\varepsilon)$, the Canonical probability distribution in non-Euclidean BG Statistical Mechanics, is given by

$$p(\varepsilon; \beta; p) = \frac{1}{A_p} \exp\left[-D_p|\beta\varepsilon - u_p|^{p-2}(\beta\varepsilon - u_p)\right], \tag{40}$$

where $D_p \equiv (\frac{x_p}{u_p})^{p-1}(p-1)^{-1}$, and the arguments x_p, u_p, A_p are given by

$$\int_0^\infty \tilde{g}(x)\, e^{-\frac{1}{p-1}|x-x_p|^{p-2}(x-x_p)}\, |x - x_p|^{p-2}(x - x_p)\, dx = 0, \tag{41}$$

$$u_p = x_p \cdot \frac{\int_0^\infty \tilde{g}(x) e^{-\frac{1}{p-1}|x-x_p|^{p-2}(x-x_p)}|x - x_p|^{p-2}dx}{\int_0^\infty \tilde{g}(x) e^{-\frac{1}{p-1}|x-x_p|^{p-2}(x-x_p)}dx}, \tag{42}$$

$$A_p = \frac{1}{\beta}\int_0^\infty \tilde{g}(x) e^{-\frac{1}{p-1}|x-x_p|^{p-2}(x-x_p)}|x - x_p|^{p-2}dx, \tag{43}$$

where we set $u_p \equiv \beta \cdot U_p$ and $x \equiv \beta_p \cdot \varepsilon$, $x_p \equiv \beta_p \cdot U_p$, and the density of states $\tilde{g}(x) \equiv g(\varepsilon = \frac{x}{\beta_p})$.

For constant density of states the probability distribution becomes

$$p(\varepsilon; \beta; p) = \beta \exp\left\{-D_p\Big[|\beta\varepsilon - 1|^{p-2}(\beta\varepsilon - 1) + 1\Big]\right\}, \tag{44}$$

with $u_p = 1$, or

$$D_p = x_p^{p-1}(p-1)^{-1}. \tag{45}$$

In non-Euclidean Tsallis Statistical Mechanics, the ordinary and escort Canonical probability distributions are respectively given by

$$p(\varepsilon; \beta_q; p, q) = \frac{1}{A_{(p,q)}} \exp_q\Big[-D_{(p,q)}|\beta_q\varepsilon - u_{(p,q)}|^{p-2}(\beta_q\varepsilon - u_{(p,q)})\Big], \tag{46}$$

$$P(\varepsilon; \beta_q; p, q) = \frac{1}{A_{(p,q)}} \exp_q{}^q\Big[-D_{(p,q)}|\beta_q\varepsilon - u_{(p,q)}|^{p-2}(\beta_q\varepsilon - u_{(p,q)})\Big], \tag{47}$$

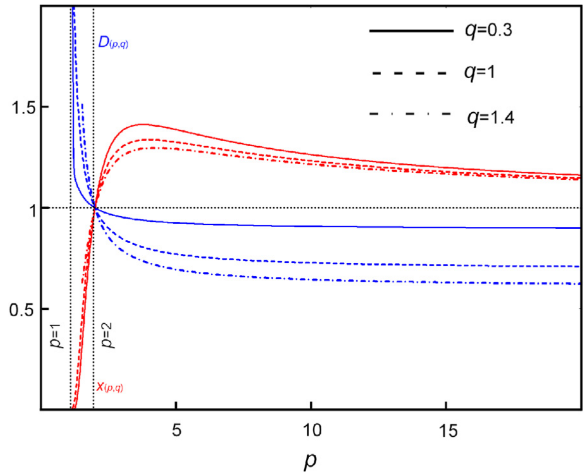

Fig. 1. The auxiliary arguments $x_{(p,q)}$ (red) and $D_{(p,q)}$ (blue) are depicted in the case of constant density of states ($g = 1$) with respect to the norm p and for the entropic indices $q = 0.3, q = 1, q = 1.4$.

where $D_{(p,q)} \equiv (\frac{x_{(p,q)}}{u_{(p,q)}})^{p-1}(p-1)^{-1}$, and the arguments $u_{(p,q)}$ and $A_{(p,q)}$ are all expressed in terms of $x_{(p,q)}$,

$$\int_0^\infty \tilde{g}(x) \exp_q{}^q \left[-\frac{1}{p-1} |x - x_{(p,q)}|^{p-2}(x - x_{(p,q)}) \right] |x - x_{(p,q)}|^{p-2}(x - x_{(p,q)}) \mathrm{d}x = 0, \tag{48}$$

$$u_{(p,q)} = x_{(p,q)} \cdot \frac{\int_0^\infty \tilde{g}(x) \exp_q{}^q \left[-\frac{1}{p-1} |x - x_{(p,q)}|^{p-2}(x - x_{(p,q)}) \right] |x - x_{(p,q)}|^{p-2} \, \mathrm{d}x}{\int_0^\infty \tilde{g}(x) \exp_q{}^q \left[-\frac{1}{p-1} |x - x_{(p,q)}|^{p-2}(x - x_{(p,q)}) \right] \, \mathrm{d}x}, \tag{49}$$

$$A_{(p,q)} = \frac{1}{\beta_q} \int_0^\infty \tilde{g}(x) \exp_q{}^q \left[-\frac{1}{p-1} |x - x_{(p,q)}|^{p-2}(x - x_{(p,q)}) \right] |x - x_{(p,q)}|^{p-2} \mathrm{d}x, \tag{50}$$

where $u_{(p,q)} \equiv \beta_q \cdot U_{(p,q)}$, $x \equiv \beta_{(p,q)} \cdot \varepsilon$, $x_{(p,q)} \equiv \beta_{(p,q)} \cdot U_{(p,q)}$. Notice that, in similar to the discrete energy states and Eq. (38), we have

$$\int_0^\infty \tilde{g}(x) \exp_q[z(x)] \mathrm{d}x = \int_0^\infty \tilde{g}(x) \exp_q{}^q[z(x)] \mathrm{d}x, \tag{51}$$

where we set $z(x) \equiv -\frac{1}{p-1} |x - x_{(p,q)}|^{p-2}(x - x_{(p,q)})$.

For arbitrary values of (p, q), the value of $x_{(p,q)}$ is numerically calculated by Eq. (48). Then, the values of $u_{(p,q)}$ and $A_{(p,q)}$ are easily obtained through Eqs. (49), (50), respectively, while the probability distributions of Eqs. (46), (47), can be explicitly expressed in terms of (p, q) and β_q.

For constant density of states the probability distributions become

$$p\,(\varepsilon; \beta_q; p, q) = \beta_q \exp_q^{-1}(D_{(p,q)}) \exp_q \left[-D_{(p,q)} |\beta_q \varepsilon - 1|^{p-2}(\beta_q \varepsilon - 1) \right], \tag{52}$$

$$P\,(\varepsilon; \beta_q; p, q) = \beta_q \exp_q{}^{-1}(D_{(p,q)}) \exp_q{}^q \left[-D_{(p,q)} |\beta_q \varepsilon - 1|^{p-2}(\beta_q \varepsilon - 1) \right], \tag{53}$$

(that is the q-deformed expression of Eq. (44)) with

$$D_{(p,q)} = x_{(p,q)}^{p-1}(p-1)^{-1}. \tag{54}$$

(For all the proofs, see: Appendix A.)

In Fig. 1 we depict both the auxiliary arguments $x_{(p,q)}$ and $D_{(p,q)}$ with respect to the norm p and for values of entropic index $q = 0.3, q = 1, q = 1.4$. Note that in order the integral in (Eq. (48)) to converge, the inequality $q < p$ must hold (see below for the proof). In Fig. 2(a)–(c) the probability distribution $\frac{1}{\beta_q} P\,(\beta_q \varepsilon; p, q)$ (Eq. (53)) is depicted as a function of $\beta_q \varepsilon$ for various values of the norm p and for the entropic indices $q = 0.3, q = 1, q = 1.4$, respectively. In contrast to the Euclidean case, either of BG statistics (Fig. 2(b)), or of Tsallis statistics ($q < 1$, Fig. 2(a), or $q > 1$, Fig. 2(c)), where the probability distribution is strictly monotonic and convex (without points of inflections), in the non-Euclidean case there are one or two points of inflection, depending on the values of (p, q). Let us first examine the non-Euclidean BG probability distribution ($q = 1$). In Fig. 2(b) we observe that in the case of super-Euclidean norms, $p > 2$, the probability distribution has two points of inflexion. The main point of inflection is located at $\beta \varepsilon = 1$, while the respective probability value $\frac{1}{\beta} p\,(\beta \varepsilon = 1; p)$ varies, depending on p. Its slope is zero, forming thus, a plateaux. The larger the value of p is, the more the plateaux is extended,

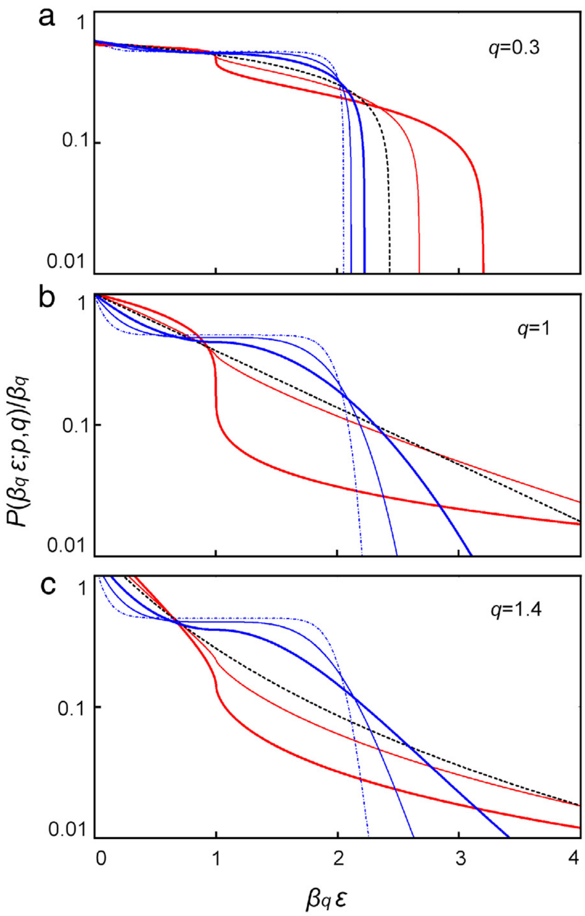

Fig. 2. The Canonical probability distribution for the continuous energy spectrum with constant density of states ($g = 1$), $\frac{1}{\beta_q} P\,(\beta_q\varepsilon;\, p,\, q)$, depicted in a semi-log scale for various sub-Euclidean ($p < 2$) or super-Euclidean norms ($p > 2$) and for the entropic indices (a) $q = 0.3$, (b) $q = 1$, (c) $q = 1.4$. In (a) we can see the cut-off condition that applies for $q < 1$, i.e., $x_{\rm Max}$ is finite, while for $q = 1$ (b) and $q > 1$ (c), the values of $\beta_q\varepsilon$ extend to infinity, i.e., $x_{\rm Max} \to \infty$. Norms used in the plots are: $p = 1.3$ (red thick solid), $p = 1.5$ (red thin solid), $p = 2$ (black dash), $p = 3$ (blue thick solid), $p = 5$ (blue thin solid), and $p = 10$ (blue dash-dot).

covering the whole interval $0 \le \beta\varepsilon \le 2$ for $p \to \infty$. Interestingly, for the limit $p \to \infty$, $\forall q$, the distribution has an upper bound for the values of energy (Appendix A), namely,

$$\frac{1}{\beta_q} P\,(\beta_q\varepsilon;\, p \to \infty,\, q) = \begin{cases} 1/2, & 0 \le \beta_q\varepsilon \le 2; \\ 0, & \beta_q\varepsilon > 2. \end{cases} \tag{55}$$

A secondary point of inflection, denoted by x_2 (not shown in Fig. 2), can be found for larger values of $\beta\varepsilon$, i.e., within the interval $1 < \beta\varepsilon$. Between the two points of inflection, the probability distribution is concave, while is convex anywhere else. In the case of sub-Euclidean norms, $p < 2$, the main point of inflection is still located at $\beta\varepsilon = 1$, but its slope is now infinite, separating the interval $x_2 < \beta\varepsilon < 1$, where the probability distribution is concave, from the interval $\beta\varepsilon > 1$, where the probability distribution is convex. The secondary point of inflection can be found only for $p \gtrsim 1.4112$. For $p \simeq 1.4112$, this point becomes $x_2 = 0$, while for smaller values of the norm $p \lesssim 1.4112$ it becomes negative ($x_2 < 0$) and disappears, and then, the whole interval $0 < \beta\varepsilon < 1$ is concave.

Similar features are found for any other values of the entropic index, either $q < 1$ (Fig. 2(a)) or $q > 1$ (Fig. 2(c)). In particular, the main point of inflection is still located at $\beta_q\varepsilon = 1$, while the secondary point of inflection can be found either for super-Euclidean or sub-Euclidean norms. For $\frac{p-2}{pq-1} > 0$, this is located in the interval $\beta_q\varepsilon > 1$, while for $\frac{p-2}{pq-1} < 0$, in the interval $\beta_q\varepsilon < 1$. In general, one can easily derive that the location of the secondary point of inflection is given by $x_2 = 1 + \mathrm{sign}(\frac{p-2}{pq-1})|\frac{(p-1)(p-2)}{pq-1}|^{\frac{1}{p-1}} \frac{1}{x_{(p,q)}}$. In Fig. 3 wedemonstrate the location of the secondary point of inflection x_2 with respect to the values (p, q). This point is observable only if is positive and smaller than the maximum cut-off value, $0 < x_2 < x_{\rm Max}$, which is given by $x_{\rm Max} = 1 + |\frac{p-1}{1-q}|^{\frac{1}{p-1}} \frac{1}{x_{(p,q)}}$ (for $q < 1$, while it is $x_{\rm Max} \to \infty$ for $q \ge 1$).

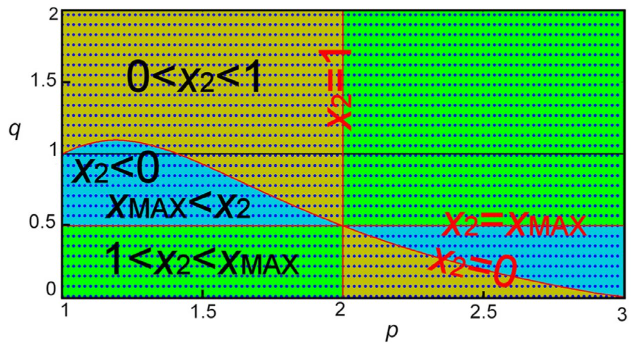

Fig. 3. The location of the secondary point of inflection with respect to the values (p, q). The indicated regions are the following: the secondary point of inflection is positive and smaller than the main one $0 < x_2 < 1$ (dark-yellow); larger than the main one and smaller than the maximum cut-off value x_{Max}, $1 < x_2 < x_{\text{Max}}$ (green); and negative or larger than x_{Max}, namely, the secondary point of inflection is not observable (light-blue).

Finally, we deal with the permissible values of the q-index with respect to a given norm p. In the case of BG Statistical Mechanics (Euclidean or not), the probability distribution decays exponentially, and thus, the relevant integrals of normalization equation (49) and of mean energy equation (51) converge for any power-like expression of the density of energy states. However, the convergence is not obvious for the non-exponential decay of Tsallis Statistical Mechanics. In this case, the integral equation (49) definitely converges for $q < 1$ because of the cut-off condition. On the contrary, for $q > 1$ and for constant density of states, the integrant at $x \to \infty$ has the asymptotic behavior x^r, with $r = \frac{p-1}{1-q}$. Then, the convergence is ensured as soon as $r < -1$, that is leading to $q < p$. For the Euclidean norm we have $q < 2$, that is the permissible values of the q-index found in Ref. [28].

3.2. Power-law density of states: the extensivity of internal energy and the equipartition of energy

Here we deal with a power-law density of energy states, namely

$$g(\varepsilon) \propto \varepsilon^{a-1}, \tag{56}$$

where $2a = f$ reads the degrees of freedom. Now Eqs. (52), (53) are written as

$$p(\varepsilon; \beta_q; p, q) g(\varepsilon) = \frac{1}{A_{(p,q)}} \exp_q\left[-D_{(p,q)} |\beta_q\varepsilon - a|^{p-2}(\beta_q\varepsilon - a)\right](\beta_q\varepsilon)^{a-1}, \tag{57}$$

$$P(\varepsilon; \beta_q; p, q) g(\varepsilon) = \frac{1}{A_{(p,q)}} {\exp_q}^{q}\left[-D_{(p,q)} |\beta_q\varepsilon - a|^{p-2}(\beta_q\varepsilon - a)\right](\beta_q\varepsilon)^{a-1}, \tag{58}$$

with

$$D_{(p,q)} \equiv \frac{1}{p-1}\left(\frac{x_{(p,q)}}{a}\right)^{p-1}, \tag{59}$$

while the arguments $x_{(p,q)}$ and $A_{(p,q)}$ can be respectively calculated by

$$\int_0^\infty {\exp_q}^{q}\left[-\frac{1}{p-1}|x - x_{(p,q)}|^{p-2}(x - x_{(p,q)})\right]|x - x_{(p,q)}|^{p-2}(x - x_{(p,q)})\, x^{a-1}dx = 0, \tag{60}$$

$$A_{(p,q)} = \frac{1}{\beta_q}\int_0^\infty {\exp_q}^{q}\left[-\frac{1}{p-1}|x - x_{(p,q)}|^{p-2}(x - x_{(p,q)})\right]|x - x_{(p,q)}|^{p-2}\, x^{a-1}dx. \tag{61}$$

The integrant Eq. (60) has the asymptotic behavior x^r for $x \to \infty$ with $r = \frac{p-1}{1-q} + a - 1$. The convergence of the integral requires $r < -1$, i.e.,

$$\frac{1}{q-1} > \frac{f/2}{p-1}, \quad \text{or,} \quad q < q_{\text{Max}} \equiv 1 + \frac{2}{f}(p-1), \tag{62}$$

where we set $a \equiv f/2$. For constant density of states ($a = 1$) we have $q < p$ (see Section 3.1), while for the 3-dimensional case ($a = \frac{3}{2}$), we have $q < \frac{5}{3} + \frac{2}{3}(p-2)$. For $p \to 1$, then $q < 1$ (i.e., $q > 1$ values are not allowed). The same result is found for $f \to \infty$ ($\forall p \geq 1$). On the other hand, for $p \to \infty$, then $q_{\text{Max}} \to \infty$ ($\forall f \leq \infty$). Therefore, the maximum possible value of the entropic index, q_{Max}, increases linearly with the increase of the p-norm, extended for the super-Euclidean norms up to $q_{\text{Max}} \to \infty$ for $p \to \infty$, while shrinking for the sub-Euclidean norms reaching $q_{\text{Max}} \to 1$ for $p = 1$.

Furthermore, the internal energy $U_{(p,q)}$ is found to be given by

$$U_{(p,q)} = a \cdot k_B T_q, \tag{63}$$

and thus, for a system of N particles with f degrees of freedom each, we have $a = \frac{Nf}{2}$, i.e.,

$$U_{(p,q)}(N) = N\frac{f}{2} \cdot k_B T_q = N \cdot U_{(p,q)}(1), \tag{64}$$

with $U_{(p,q)}(1) = \frac{f}{2} \cdot k_B T_q$. Eq. (64) reads the equipartition of energy in each degree of freedom and the extensivity of internal energy ($U_{(p,q)}(N) \propto N$) under the well-defined definition of physical temperature T_q.

It is interesting that even in the non-Euclidean adaptation of Tsallis Statistical Mechanics, the physical temperature T_q coincides with the kinetically defined temperature T_K [14], that is

$$U_{(p,q)} \equiv \frac{3}{2} \cdot k_B T_K, \tag{65}$$

in the 3-dimensional case. Then, given of $U_{(p,q)} = \frac{3}{2} \cdot k_B T_q$, we derive $T_K = T_q$. (Note that the Lagrangian temperature, $T \equiv -\frac{1}{k_B \lambda_2}$, is mostly used in the (Euclidean or non-Euclidean) BG Statistical Mechanics ($q \to 1$) and coincides with the physical temperature T_q. Namely, $U_p(N) = N\frac{f}{2} \cdot k_B T = N \cdot U_p(1)$ with $U_p(1) = \frac{f}{2} \cdot k_B T$.)

The quantity $\kappa \equiv 1/(q-1)$ is called kappa index, and the transformed escort probability distribution is called kappa distribution, widely known in space and plasma physics [21]. According to (62), we have $\kappa > f_p/2$, where $f_p \equiv f/(p-1)$ represents the effective degrees of freedom (see also Ref. [23]). Finally, the non-Euclidean-normed kappa distribution is given in terms of the modified kappa index $\kappa_0 \equiv \kappa - f_p/2 \in (0, \infty)$,

$$P(\varepsilon; T_q; p, \kappa_0) = \frac{(k_B T_q/x_*)^{-f/2}}{A_{(p,\kappa_0)}} \left\{ 1 + \frac{1}{\kappa_0(p-1)} \left[\frac{f}{2} + x_*^{p-1} \left| \frac{\varepsilon}{k_B T_q} - \frac{f}{2} \right|^{p-2} \left(\frac{\varepsilon}{k_B T_q} - \frac{f}{2} \right) \right] \right\}^{-\kappa_0 - 1 - \frac{f/2}{p-1}} \varepsilon^{f/2-1}, \tag{66}$$

where at the limit of high energy becomes

$$P(\varepsilon; T_q; p, \kappa_0) \cong \frac{(k_B T_q/x_*)^{-f/2}}{A_{(p,\kappa_0)}} \left[1 + \frac{x_*^{p-1}}{\kappa_0(p-1)} \left(\frac{\varepsilon}{k_B T_q} \right)^{p-1} \right]^{-\kappa_0 - 1 - \frac{f/2}{p-1}} \varepsilon^{f/2-1}. \tag{67}$$

The argument $x_* = x_*(p, \kappa_0)$ is implicitly given by

$$\int_0^\infty \left\{ 1 + \frac{1}{\kappa_0(p-1)} \left[\frac{f}{2} + \left| x - x_* \frac{f}{2} \right|^{p-2} \left(x - x_* \frac{f}{2} \right) \right] \right\}^{-\kappa_0 - 1 - \frac{f/2}{p-1}} \left| x - x_* \frac{f}{2} \right|^{p-2} \left(x - x_* \frac{f}{2} \right) x^{f/2-1} \mathrm{d}x = 0, \tag{68}$$

with $x_* = 1$ for $p = 2$ (for any κ_0), and the normalization constant becomes

$$A_{(p,\kappa_0)} = \int_0^\infty \left\{ 1 + \frac{1}{\kappa_0(p-1)} \left[\frac{f}{2} + \left| x - x_* \frac{f}{2} \right|^{p-2} \left(x - x_* \frac{f}{2} \right) \right] \right\}^{-\kappa_0 - 1 - \frac{f/2}{p-1}} x^{f/2-1} \mathrm{d}x = 0. \tag{69}$$

We repeat that T_q denotes the actual temperature of the system. As it has been shown in detail in the theory of kappa distributions (e.g., Refs. [14,21,23]), T_q is not dependent on q or $\kappa = 1/(q-1)$, but is one of the independent parameters of the distribution.

4. Thermodynamics

The connection with thermodynamics arises through the known classical formulations, such as

$$U_p = -\frac{\partial \ln \mathcal{Z}_p}{\partial \beta}, \tag{70}$$

$$S^{\mathcal{BG}} = k_B \ln \mathcal{Z}_p + \frac{U_p}{T}, \tag{71}$$

$$\frac{1}{T} = \frac{\partial S^{\mathcal{BG}}}{\partial U_p}, \tag{72}$$

$$F_p \equiv U_p - TS^{\mathcal{BG}} = -k_B T \ln \mathcal{Z}_p, \tag{73}$$

for the non-Euclidean BG statistical description (where we restore the Boltzmann's constant k_B), and

$$U_{(p,q)} = -\frac{\partial \ln_q \mathcal{Z}_{(p,q)}}{\partial \beta}, \tag{74}$$

$$S^{\mathcal{TS}} = k_B \ln_q \mathcal{Z}_{(p,q)} + \frac{U_{(p,q)}}{T}, \tag{75}$$

$$\frac{1}{T} = \frac{\partial S^{\mathcal{TS}}}{\partial U_{(p,q)}}, \tag{76}$$

$$F_{(p,q)} \equiv U_{(p,q)} - TS^{\mathcal{TS}} = -k_B T \ln_q \mathcal{Z}_{(p,q)}, \tag{77}$$

for the non-Euclidean Tsallis Statistical description, recovering the preceded relations for $q \to 1$. (For the proves, see: Appendix B.) It is remarkable that the relations connecting Tsallis Statistical Mechanics with Thermodynamics are preserved as in the Euclidean case, that is simply by considering the formalism of the q-deformed logarithm. Similarly, the concept of physical temperature T_q remains under the Euclidean definition of Eq. (14), that is

$$T_q \equiv T \cdot \phi_q \Rightarrow T_q \equiv \left(\frac{\partial S^{\mathcal{TS}}}{\partial U_{(p,q)}}\right)^{-1} \left[1 + \frac{1}{k_B} S^{\mathcal{TS}} (1-q)\right]. \tag{78}$$

5. Conclusions

This analysis introduced a possible generalization of Statistical Mechanics, realized within the framework of non-Euclidean metrics induced by the $\mathcal{L}^p$ norms. In particular, the concept of internal energy was generalized so as to be interpreted by the non-Euclidean $\mathcal{L}^p$-normed expectation value of a given spectrum of energy states $\{\varepsilon_k\}_{k=1}^W$. Therefore, the presented non-Euclidean adaptation of Statistical Mechanics involves extremizing an entropic formulation under the constraint of the internal energy, being interpreted by the non-Euclidean $\mathcal{L}^p$-normed expectation value of energy spectrum (Canonical Ensemble). In this way, the extracted stationary probability distribution characterizes the most generalized formulation of a Canonical probability distribution.

Two entropic formulations were adopted; the classical Boltzmann–Gibbs entropy $S^{\mathcal{BG}}$ and its generalization $S^{\mathcal{TS}}$ that was introduced by Tsallis [8] and considered in a numerous applications since today (e.g., see: Refs. [24,29]). The former is related to the internal energy being interpreted simply by the ordinary expectation value of the energy spectrum $\{\varepsilon_k\}_{k=1}^W$, i.e., $U = \langle \varepsilon \rangle$. However, the latter is totally interwoven with the concept of the escort or q-expectation values, and thus, the internal energy is interpreted by the escort expectation value of $\{\varepsilon_k\}_{k=1}^W$, that is $U_q = \langle \varepsilon \rangle_q$.

Therefore, the non-Euclidean BG and Tsallis Statistical Mechanics involves finding the relevant non-Euclidean adaptation of the Canonical probability distribution by respectively extremizing the BG and Tsallis entropy under the appropriate non-Euclidean constraint of internal energy, which is denoted respectively by $U_p = \langle \varepsilon \rangle_p$ and $U_{(p,q)} = \langle \varepsilon \rangle_{(p,q)}$.

This is constructed by means of the non-Euclidean norm operator, respectively denoted by $\hat{\mathcal{L}}_p$ and $\hat{\mathcal{L}}_{(p,q)}$, which helps to automatically retrieve the non-Euclidean representation of a formulation from its respective Euclidean one. In this way, the non-Euclidean $\mathcal{L}^p$-normed expectation value of the energy spectrum $\{\varepsilon_k\}_{k=1}^W$ is considered as the Euclidean mean of $\{\hat{\mathcal{L}}_p(\varepsilon_k)\}_{k=1}^W$, namely, $U_p = \langle \varepsilon \rangle_p = \langle \hat{\mathcal{L}}_p(\varepsilon) \rangle$. In similar, the non-Euclidean $\mathcal{L}^p$-normed q-expectation value of the energy spectrum $\{\varepsilon_k\}_{k=1}^W$ is considered as the Euclidean escort mean of $\{\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k)\}_{k=1}^W$, namely, $U_{(p,q)} = \langle \varepsilon \rangle_{(p,q)} = \langle \hat{\mathcal{L}}_{(p,q)}(\varepsilon) \rangle_q$.

Given the set of states $\{y_k\}_{k=1}^W$ of a physical quantity Y that is associated with the probability distribution $\{p_k\}_{k=1}^W$, being included in the Euclidean representation of a formulation, i.e., $F[\{y_k\}_{k=1}^W; \{p_k\}_{k=1}^W]$, we can very often automatically retrieve the respective non-Euclidean representation by replacing y_k with $\hat{\mathcal{L}}_p(y_k)$, $\forall\, k = 1, \ldots, W$, i.e., $F_p[\{y_k\}_{k=1}^W; \{p_k\}_{k=1}^W] = F[\{\hat{\mathcal{L}}_p(y_k)\}_{k=1}^W; \{p_k\}_{k=1}^W]$. On the other hand, the respective (Euclidean) escort representation is known to be derived by replacing p_k with P_k, $\forall\, k = 1, \ldots, W$, i.e., $F_q[\{y_k\}_{k=1}^W; \{p_k\}_{k=1}^W] = F[\{y_k\}_{k=1}^W; \{P_k\}_{k=1}^W]$. Finally, the interwoven non-Euclidean escort representation is given by replacing y_k with $\hat{\mathcal{L}}_{(p,q)}(y_k)$ and p_k with P_k, $\forall\, k = 1, \ldots, W$, i.e., $F_{(p,q)}[\{y_k\}_{k=1}^W; \{p_k\}_{k=1}^W] = F[\{\hat{\mathcal{L}}_{(p,q)}(y_k)\}_{k=1}^W; \{P_k\}_{k=1}^W]$.

The non-Euclidean Canonical probability distributions, as derived from the BG and Tsallis statistics, are respectively given by an exponential, and by a q-deformed exponential, of a power-law dependence on energy states. Both of these non-Euclidean expressions can be derived from the respective Euclidean one, in which the non-Euclidean norm operator is acting to the energy states. Namely, the kth energy state ε_k ($\forall\, k = 1, \ldots, W$) is replaced by $\hat{\mathcal{L}}_p(\varepsilon_k)$ and $\hat{\mathcal{L}}_{(p,q)}(\varepsilon_k)$, respectively for BG and Tsallis statistics.

Furthermore, the case of the continuous energy spectrum was thoroughly examined. In particular, for a power-law density of energy states, the non-Euclidean Canonical probability was analytically derived, for both the non-Euclidean adaptations of BG and Tsallis Statistical Mechanics. In the case of constant density of energy states, the distributions found to be given by a simple expression. The relevant non-Euclidean-normed kappa distributions were derived.

The statistical analysis of systems exhibiting non-extensive behavior, among others (e.g., see Refs. [30,31]), involves fitting the empirical distributions of physical quantities x with the bi-parametrical function $p(x) \propto (1+ax)^{-b}$, where the inverse temperature parameter β_q and the entropic index q are respectively calculated by the estimated fitting parameters $a = a(\beta_q; q)$ and $b = q/(q-1)$ (e.g., see Ref. [32]). Moreover, in non-Euclidean statistics, a tri-parametrical function has to be utilized, namely, $p(x) \propto (1 + a|x_c|^{p-2}x_c)^{-b}$, expressed in terms of the centered variable $x_c \equiv x - \langle x\rangle_{(p,q)}$, or even, $p(x) \propto (1 + a|x|^{p-1})^{-b}$ (for large values of x, so that $x \gg \langle x\rangle_{(p,q)}$), where the third parameter determines the value of the p-norm.

The equipartition of energy in each degree of freedom f and the extensivity of internal energy, were shown. Remarkably, even in the non-Euclidean adaptation of Tsallis Statistical Mechanics, the physical temperature T_q keeps playing the role of the kinetically defined temperature T_K, in similar for the stationary states in the Euclidean case [14].

The permissible values of the q-index depend on the given p-norm. In particular, in order the internal energy to be interpreted as the $\mathcal{L}^p$-normed q-expectation value of the energy spectrum, the convergence of the relevant integral is required, that is leading to $q < \frac{2}{f}(p-1)+1$. In this way, the maximum possible value of the entropic index q increases linearly with the increase of the p-norm.

It was shown that the connection with thermodynamics, remarkably, arises through the well-known classical formalisms. In these relations, the partition function and the internal energy are replaced by the respective non-Euclidean one. The relevant relations connecting Tsallis Statistical Mechanics with Thermodynamics are preserved as in the Euclidean case, that is simply by considering the formalism of the q-deformed logarithm. Similarly, the physical temperature T_q remains under its Euclidean definition.

Statistical Mechanics is used to determine how particle systems behave when they reside at thermal equilibrium. These systems are described by Boltzmann–Gibbs Statistical Mechanics, where the distribution of particle velocities in the absence of potential energy is given by a Maxwellian distribution. Space plasmas are exotic particle systems, which are better described by kappa distributions rather than Maxwellians [21,33]. In general, the larger the value of the kappa index, the closer the plasma is to thermal equilibrium. When the kappa parameter reaches infinity, the plasma is exactly at thermal equilibrium and the distribution of space plasma is reduced to a Maxwellian. In this way, the kappa parameter is a novel thermal observable (like temperature, density, pressure, etc.) which can tell us about the “thermodynamic distance” of a system from thermal equilibrium [34]. The introduced non-Euclidean kappa distribution has one more parameter characterizing the system, that is, the norm parameter p. There have been already reported and published several cases of space plasmas, where the particle populations are better described by the non-Euclidean kappa distribution (Eq. (67)) rather than the standard kappa distributions. For example, see the model of Refs. [35,36] used to describe the “Lion Roars” observed in the Magnetosheath, and the solar wind observations of the Cluster spacecraft. Very important are the simulations of Ref. [37] in the regime of weakly/strongly coupled space plasmas, where the show numerically the possible existence of such a distribution. Finally, we mention that the work presented here was first announced in the SigmaPhi-2008 Conference on Statistical Mechanics [38], where several physical applications have been noted, such as, in chemical kinetics by generalizing the Arrhenius equation, and the glass-forming materials.

The classical Statistical Mechanics has stood the test of time for describing Earthy systems which reside at thermal equilibrium and are aligned to Maxwellian distributions. Space plasmas from the solar wind to planetary magnetospheres and the outer heliosphere are systems out of thermal equilibrium, are connected with the more general statistical framework of non-extensive statistical mechanics [29]. Now that the theory of kappa distributions is complete for describing particle systems of any $\mathcal{L}^p$ norms, the full strength and capability of these distributions and their statistical background are available for the physics community to study and improve our understanding of space plasmas and other systems with similar statistical behavior.

Appendix A. Canonical probability distribution for continuous energy spectrum

In the case of continuous energy spectrum and within the framework of non-Euclidean Tsallis Statistical Mechanics, the Canonical probability distribution is given by Eqs. (46), (47). For a power-law density $g(\varepsilon) \propto \varepsilon^{a-1}$ (Eq. (56)), or $\tilde{g}(x) \equiv g(\varepsilon = \frac{x}{\beta_{(p,q)}}) \propto x^{a-1}$, the probability distribution becomes

$$p\,(\varepsilon;\,\beta_q;\,p,\,q)\,g(\varepsilon) = \frac{1}{A_{(p,q)}}\exp_q\Big[-D_{(p,q)}|\beta_q\varepsilon - u_{(p,q)}|^{p-1}\mathrm{sign}(\beta_q\varepsilon - u_{(p,q)})\Big]\varepsilon^{a-1}, \tag{79}$$

where $D_{(p,q)} \equiv (\frac{x_{(p,q)}}{u_{(p,q)}})^{p-1}(p-1)^{-1}$, while the dimensionless terms $x_{(p,q)}$, $u_{(p,q)}$, and $\beta_q A_{(p,q)}$, depend only on p, q and a, and are respectively given by

$$\int_0^\infty \exp_q{}^q\Big[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\Big]|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\,x^{a-1}\mathrm{d}x = 0, \tag{80}$$

$$u_{(p,q)} = x_{(p,q)}\cdot\frac{\int_0^\infty \exp_q{}^q\left[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\right]|x-x_{(p,q)}|^{p-2}\,x^{a-1}\mathrm{d}x}{\int_0^\infty \exp_q\left[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\right]x^{a-1}\mathrm{d}x}, \tag{81}$$

$$A_{(p,q)} = \frac{1}{\beta_q}\int_0^{\infty} \exp_q{}^q\left[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\right]|x-x_{(p,q)}|^{p-2}\,x^{a-1}\mathrm{d}x. \tag{82}$$

Eqs. (80)–(82) are deduced by setting $x \equiv \beta_{(p,q)}\varepsilon$, $x_{(p,q)} \equiv \beta_{(p,q)}U_{(p,q)}$, and $u_{(p,q)} \equiv x_{(p,q)}\tilde{\phi}_{(p,q)} = \beta_q U_{(p,q)}$ with $\tilde{\phi}_{(p,q)} \equiv \frac{\beta_q}{\beta_{(p,q)}} = \beta_{(p,q)}{}^{p-2}\phi_{(p,q)}$.

Then, we easily show that

$$u_{(p,q)} = a, \tag{83}$$

Indeed, starting from Eq. (80), we have

$$\begin{aligned} 0 &= \int_0^{\infty} \exp_q{}^q\Big[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\Big]|x-x_{(p,q)}|^{p-2}(x^a - x_{(p,q)}x^{a-1})\,\mathrm{d}x \\ &= \int_0^{\infty} \exp_q{}^q\Big[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\Big]|x-x_{(p,q)}|^{p-2}x^a\,\mathrm{d}x - x_{(p,q)}\beta_q A_{(p,q)}, \end{aligned}$$

while the integral is equal to

$$\begin{aligned} &-\int_0^{\infty} x^a\,\mathrm{d}\exp_q\Big[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\Big] \\ &= a\cdot\int_0^{\infty}\exp_q\Big[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\Big]x^{a-1}\,\mathrm{d}x = a\cdot\beta_q A_{(p,q)}\cdot\frac{x_{(p,q)}}{u_{(p,q)}}. \end{aligned}$$

The integration by parts gave

$$\Big\{x^a\exp_q\Big[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\Big]\Big\}\Big|_0^{\infty} = 0,$$

because of $a > 0$ (for the lower boundary) and $\frac{1}{q-1} > \frac{a}{p-1}$ (Eq. (62)) (for the upper boundary). Hence, we conclude in Eq. (83). (It would be interesting to connect the non-Euclidean moments of Tsallis Canonical Probability distribution with Beta functions as in the Euclidean case [14].)

Specifically for the case $a = 1$, that is for constant density of states, we have $u_{(p,q)} = 1$ and

$$\begin{aligned} \beta_q A_{(p,q)} &= \int_{-\frac{1}{p-1}x_{(p,q)}^{p-1}}^{\infty}\mathrm{d}\exp_q\Big[-\frac{1}{p-1}|x-x_{(p,q)}|^{p-2}(x-x_{(p,q)})\Big] \\ &= \exp_q\left(D_{(p,q)}\right). \end{aligned} \tag{84}$$

Therefore, we conclude in the probability distribution of Eqs. (52), (53), and thus, in Eq. (44) for $q \to 1$. (Note that for $q < 1$, the upper boundary of all the above integrals becomes equal to $\frac{1}{1-q}$, instead of infinity because of the Tsallis cut-off condition [28]).

Finally we show the probability distribution at the limit $p \to \infty$, $\forall\ q$, given by Eq. (55). The q-deformed exponential factor in Eq. (53) becomes zero for $|x - x_{(p,q)}| > 1$ and unity for $|x - x_{(p,q)}| \le 1$. We justify this statement as follows: The argument $-\frac{1}{p-1}|x-x_{(p,q)}|^{p-1}\mathrm{sign}(x-x_{(p,q)})$ becomes $-\infty$ and 0, respectively. Hence, for $q = 1$ and the relevant exponential function, the statement is trivial. In addition, it is apparent that, for $|x - x_{(p,q)}| \le 1$ where the mentioned argument is zero, we have the q-deformed exponential factor to become unity $\forall q \ne 1$. However, for $|x - x_{(p,q)}| > 1$ where the argument is $-\infty$, the q-deformed exponential factor $\exp_q(-\infty) = [1 + (1-q)(-\infty)](\frac{1}{1-q})$ vanishes for $q < 1$ due to the cut-off condition, while for $q > 1$ due to the negative sign of the power $\frac{1}{1-q}$. Thus, by setting $x_{(p,q)} = 1$, the relevant integral of Eq. (49) (for $\tilde{g}(x) \to 1$) is restricted to the interval $0 \le x \le 2$ and equals to zero. Hence, $x_{(p,q)} = 1$ satisfies Eq. (49), and, given the uniqueness of $\mathcal{L}^p$-normed means, we conclude that $x_{(p,q)} = 1$ is the only possible solution. Similarly, we conclude that the probability $p(\beta_q\varepsilon; p \to \infty, q)$ is non-zero only within the interval $|\beta_q\varepsilon - 1| \le 1$, or $0 \le \beta_q\varepsilon \le 2$, hence $A_{(p,q)} = \frac{1}{2}$.

Appendix B. Analytical calculation of thermodynamic relations

We rewrite the action of the non-Euclidean norm operator (Eq. (5)) as

$$\hat{\mathcal{L}}_p(\varepsilon_k) = C\cdot|\varepsilon_k - U_p|^{p-1}\mathrm{sign}(\varepsilon_k - U_p) + U_p, \tag{85}$$

with the argument C given by

$$C(\{\varepsilon_k, p_k\}_{k=1}^{W}; p) = \frac{1}{(p-1)\phi_p}. \tag{86}$$

Then, starting from $\mathcal{Z}_p(\beta) = \sum_{k=1}^{W} \exp\left[-\beta \hat{\mathcal{L}}_p(\varepsilon_k)\right]$, we have

$$-\frac{\partial \ln \mathcal{Z}_p}{\partial \beta} = \frac{\partial (C\beta)}{\partial \beta} \sum_{k=1}^{W} p_k |\varepsilon_k - U_p|^{p-1} \mathrm{sign}(\varepsilon_k - U_p) + \left[1 - C(p-1) \sum_{k=1}^{W} p_k |\varepsilon_k - U_p|^{p-2}\right] \beta \frac{\partial U_p}{\partial \beta} + U_p = \left[1 - C(p-1)\phi_p\right] \beta \frac{\partial U_p}{\partial \beta} + U_p,$$

(because of Eq. (4)), and thus, given Eqs. (86), (7), we conclude in

$$U_p = -\frac{\partial \ln \mathcal{Z}_p}{\partial \beta}.$$

In particular, the specific expression of the argument C as given in Eq. (86) leads to both the thermodynamic relation of Eq. (74) and the zero-mean property of the non-Euclidean norm operator (Eq. (8)), according to the scheme

$$U_p = -\frac{\partial \ln \mathcal{Z}_p}{\partial \beta} \Leftrightarrow C = \frac{1}{(p-1)\phi_p} \Leftrightarrow \sum_{k=1}^{W} p_k \frac{\partial}{\partial \beta} \hat{\mathcal{L}}_p(\varepsilon_k) = 0. \tag{87}$$

We proceed in the proof of the relevant relation Eq. (74) for the non-Euclidean Tsallis thermodynamics. Starting from Eq. (20) we have

$$\phi_q = \mathcal{Z}_{(p,q)}^{1-q} \exp_q{}^{q-1}(-\beta_q U_{(p,q)}), \tag{88}$$

from which we calculate the derivative

$$\frac{\partial \phi_q}{\partial \beta_q} = (1-q)\mathcal{Z}_{(p,q)}^{-q} \frac{\partial \mathcal{Z}_{(p,q)}}{\partial \beta_q} \exp_q{}^{q-1}(-\beta_q U_{(p,q)}) + (1-q)\mathcal{Z}_{(p,q)}^{1-q} \exp_q{}^{2q-2}(-\beta_q U_{(p,q)}) \left(\beta_q \frac{\partial U_{(p,q)}}{\partial \beta_q} + U_{(p,q)}\right). \tag{89}$$

Then, we calculate the derivative,

$$\frac{\partial \beta}{\partial \beta_q} = \mathcal{Z}_{(p,q)}^{1-q} \exp_q{}^{q-1}(-\beta_q U_{(p,q)}) + (1-q)\beta_q \mathcal{Z}_{(p,q)}^{-q} \frac{\partial \mathcal{Z}_{(p,q)}}{\partial \beta_q} \exp_q{}^{q-1}(-\beta_q U_{(p,q)}) + (1-q)\beta_q \mathcal{Z}_{(p,q)}^{1-q} \exp_q{}^{2q-2}(-\beta_q U_{(p,q)}) \left(\beta_q \frac{\partial U_{(p,q)}}{\partial \beta_q} + U_{(p,q)}\right), \tag{90}$$

where we used Eqs. (88), (89). Starting from Eq. (23), we find

$$-\mathcal{Z}_{(p,q)}^{-q} \frac{\partial \mathcal{Z}_{(p,q)}}{\partial \beta_q} = \mathcal{Z}_{(p,q)}^{1-q} \exp_q{}^{q-1}(-\beta_q U_{(p,q)}) \left(\beta_q \frac{\partial U_{(p,q)}}{\partial \beta_q} + U_{(p,q)}\right) - \beta_q \frac{\partial U_{(p,q)}}{\partial \beta_q} \mathcal{Z}_{(p,q)}^{1-q}. \tag{91}$$

By substituting the quantity $-\mathcal{Z}_{(p,q)}^{-q} \frac{\partial \mathcal{Z}_{(p,q)}}{\partial \beta_q}$ of Eq. (91) in Eq. (90), we derive

$$\frac{\partial \beta}{\partial \beta_q} = \mathcal{Z}_{(p,q)}^{1-q} \exp_q{}^{q-1}(-\beta_q U_{(p,q)}) \left[1 + (1-q){\beta_q}^2 \frac{\partial U_{(p,q)}}{\partial \beta_q}\right]. \tag{92}$$

Moreover, Eq. (91) is rewritten as

$$-\mathcal{Z}_{(p,q)}^{-q} \frac{\partial \mathcal{Z}_{(p,q)}}{\partial \beta_q} = \mathcal{Z}_{(p,q)}^{1-q} \exp_q{}^{q-1}(-\beta_q U_{(p,q)}) U_{(p,q)} \left[1 + (1-q){\beta_q}^2 \frac{\partial U_{(p,q)}}{\partial \beta_q}\right]. \tag{93}$$

Then, by combining Eqs. (92) and (93), we have

$$-\mathcal{Z}_{(p,q)}^{-q} \frac{\partial \mathcal{Z}_{(p,q)}}{\partial \beta} = -\frac{\partial}{\partial \beta}\left(\frac{1 - \mathcal{Z}_{(p,q)}^{1-q}}{q-1}\right) = U_{(p,q)},$$

from which, we derive Eq. (74).

Furthermore, we prove Eq. (75). Indeed, starting from Eq. (88), we have

$$\mathcal{Z}_{(p,q)}^{1-q} = \phi_q \exp_q{}^{1-q}(-\beta_q U_{(p,q)}) = \phi_q \left[1 - (1-q)\beta_q U_{(p,q)}\right] \Rightarrow \frac{1-\phi_q}{q-1} = \frac{1 - \mathcal{Z}_{(p,q)}^{1-q}}{q-1} + \phi_q \beta_q U_{(p,q)} \Rightarrow S^{\mathcal{TS}} = k_B \ln_q \mathcal{Z}_{(p,q)} + \frac{U_{(p,q)}}{T}, \tag{94}$$

while the Helmholtz free energy $F_{(p,q)} \equiv U_{(p,q)} - TS^{\mathcal{T}\mathcal{S}}$ is given by

$$F_{(p,q)} = -T\left(S^{\mathcal{T}\mathcal{S}} - \frac{U_{(p,q)}}{T}\right) = -k_B T \ln_q \mathcal{Z}_{(p,q)}, \tag{95}$$

which reads Eq. (77). Finally, Eq. (94) readily leads to Eq. (75) by calculating $\frac{\partial S^{\mathcal{T}\mathcal{S}}}{\partial U_{(p,q)}} = \frac{\partial S^{\mathcal{T}\mathcal{S}}}{\partial \beta} / \frac{\partial U_{(p,q)}}{\partial \beta}$. (Note that Eqs. (70)–(73) can be derived, either by starting from the BG formulation, or by setting $q \to 1$ in Eqs. (74)–(77).)

References

[1] G. Livadiotis, Entropy 14 (2012) 2375.
[2] G. Livadiotis, Physica A 375 (2007) 518.
[3] G. Livadiotis, Physica A 387 (2008) 2471.
[4] C. Tsallis, R.S. Mendes, A.R. Plastino, Physica A 261 (1998) 534.
[5] C. Beck, F. Schlogl, Thermodynamics of Chaotic Systems, Cambridge University Press, Cambridge, 1993.
[6] L.P. Chimento, F. Pennini, A. Plastino, Phys. Rev. E 62 (2000) 7462.
[7] J. Naudts, JIPAM 5 (2004) 112.
[8] C. Tsallis, J. Stat. Phys. 52 (1988) 479.
[9] R. Silva, A.R. Plastino, J.A.S. Lima, Phys. Lett. A 249 (1998) 401–408.
[10] T. Yamano, Physica A 305 (2002) 486.
[11] S. Abe, Physica A 269 (1999) 403.
[12] S.K. Rama, Phys. Lett. A 276 (2000) 103.
[13] G. Livadiotis, Entropy 16 (2014) 4290.
[14] G. Livadiotis, D.J. McComas, J. Geophys. Res. 114 (2009) A11105.
[15] A. Baranyai, Phys. Rev. E 61 (2000) R3306.
[16] A. Baranyai, Phys. Rev. E 62 (2000) 5989.
[17] S. Abe, S. Martinez, F. Pennini, A. Plastino, Phys. Lett. A 281 (2001) 126.
[18] Q.A. Wang, L. Nivanen, A. Le Mehaute, M. Pezeril, J. Phys. A 35 (2002) 7003.
[19] R. Toral, Physica A 317 (2003) 209.
[20] W.G. Hoover, C.G. Hoover, Phys. Rev. E 77 (2008) 041104.
[21] G. Livadiotis, J. Geophys. Res. 120 (2015) 1607.
[22] G. Livadiotis, J. Geophys. Res. 120 (2015) 880.
[23] G. Livadiotis, Entropy 17 (2015) 2062.
[24] C. Tsallis, Braz. J. Phys. 29 (1999) 1.
[25] M. Gell-Mann, C. Tsallis (Eds.), Nonextensive Entropy-Interdisciplinary Applications, Oxford Univ. Press, 2004.
[26] A.R. Lima, T.J.P. Penna, Physica A 265 (1999) 221.
[27] G. Livadiotis, J. Math. Chem. 45 (2009) 930.
[28] G.L. Ferria, S. Martínez, A. Plastino, Physica A 347 (2005) 205.
[29] C. Tsallis, Introduction to Nonextensive Statistical Mechanics, Springer, New York, 2009.
[30] C. Tsallis, F.C. Sá Barreto, E.D. Loh, Phys. Rev. E 52 (1995) 1447.
[31] A.R. Plastino, A. Plastino, H. Vucetich, Phys. Lett. A 207 (1995) 42.
[32] D.H. Zanette, M.A. Montemurro, Phys. Lett. A 324 (2004) 383.
[33] G. Livadiotis, D.J. McComas, Space Sci. Rev. 75 (2015) 183.
[34] G. Livadiotis, D.J. McComas, Phys. Scr. 82 (2010) 035003.
[35] M.N. Qureshi, et al. In: M. Velli, R. Vruno, F. Malara (Eds.), Proc. Solar Wind Conf. 10, 2003, p. 489.
[36] M.N. Qureshi, W. Masood, H.A. Shah, P.H. Yoon, S.J. Schwartz, In: AGU Fall Meeting, 2013.
[37] B.M. Randol, E.R. Christian, J. Geophys. Res. 119 (2014) 7025.
[38] G. Livadiotis, In: 2nd SigmaPhi Conf. on Statistical Mechanics, 2008. http://www.sigmaphi.polito.it/2008/Abstract_pdf.